TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 9, Pages 3493–3508
S 0002-9947(04)03533-0
Article electronically published on September 2, 2004

FIXED POINT INDEX IN SYMMETRIC PRODUCTS

JOSÉ M. SALAZAR

ABSTRACT. Let U be an open subset of a locally compact metric ANR X and let $f : U \to X$ be a continuous map. In this paper we study the fixed point index of the map that f induces in the n-symmetric product of X, $F_n(X)$. This index can detect the existence of periodic orbits of period $\leq n$ of f, and it can be used to obtain the Euler characteristic of the n-symmetric product of a manifold X, $\chi(F_n(X))$. We compute $\chi(F_n(X))$ for all orientable compact surfaces without boundary.

1. Introduction

Let X be a locally compact metric ANR, $f : U \subset X \to X$ a semidynamical system and $K \subset U$ a compact isolated invariant set with respect to f. In this paper we construct the fixed point index of the map that f induces in the spaces $F_n(X)$ of the non-empty finite subsets of X with at most n elements, endowed with the Hausdorff metric. These spaces were defined in 1931 by Borsuk and Ulam, in [3], with the name of n-symmetric product of X. They studied some topological properties which X induces in $F_n(X)$ and topological properties of the space $F_n([0, 1])$.

Our fixed point index detects the existence of periodic orbits of f in K of period less than or equal to n.

Let 2^X be the hyperspace of all non-empty compact subsets of X endowed with the Hausdorff metric d_H, defined by

$$d_H(C, D) = inf\{\epsilon > 0 : C \subset B(D, \epsilon) \text{ and } D \subset B(C, \epsilon)\},$$

and let $C_n(X) \subset 2^X$ be the hyperspace of all non-empty compact subsets of X having at most n connected components. Our study will be harder than the analysis of the fixed point indices constructed in [23] for the hyperspaces 2^X and $C_n(X)$. The difficulties follow from the fact that the topological structure of $F_n(X)$ is more complicated than that of 2^X and $C_n(X)$.

In Section 2 we prove that our construction is consistent and we show the most important properties. We also compute the index for $K = \{p\}$ a non-attracting and non-repelling fixed point of a local homeomorphism f of $\mathbb{R}^2$.

If f is an orientation-preserving local homeomorphism of the plane and $\{p\}$ is a fixed point of f that is an isolated invariant set which is not an attractor nor

Received by the editors May 23, 2003 and, in revised form, October 22, 2003.
2000 *Mathematics Subject Classification.* Primary 54H20, 54H25.
Key words and phrases. Fixed point index, hyperspaces, symmetric product, semidynamical systems.

a repeller, Le Calvez and Yoccoz proved, in [14], that there exist integers $r, q \geq 1$ such that the fixed point index

$$i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}. \end{cases}$$

In the above setting we will show that the fixed point index of the map that f induces in $F_n(\mathbb{R}^2)$ at $\{p\}$ is

$$\frac{\sum_{j=1}^{n} i_{\mathbb{R}^2}(f^j, p)}{n}$$

for every $n \leq r$. We will give the proof of this result in Section 4.

In Section 3 we give techniques for computing the Euler characteristic of $F_n(X)$, for X a finite dimensional manifold.

Sometimes the spaces $F_n(X)$ are topologically equivalent to convex subsets of a euclidean space. In this case our results have similarities with the computations given in [7].

Although many authors have considered the study of the topological structure of the spaces $F_n(X)$, the topological characterizations are exceptional. Borsuk and Ulam, in [3], proved that $F_n([0,1]) \simeq [0,1]^n$ for $n \leq 3$. Borsuk, in [2], claimed that $F_3(S^1) \simeq S^1 \times S^2$ but Bott, in [4], showed that $F_3(S^1) \simeq S^3$. Molski, in [18], saw that $F_2([0,1]^2) \simeq [0,1]^4$. In the same direction we have the work of Schori, [26], where there is a characterization of spaces of the type $F_n([0,1]^m)$, obtained by using suitable equivalence relations. Likewise we have some results about the topological properties of the symmetric products. In [3] it is proved that $dim(F_n([0,1])) = n$ for all n and that $F_n([0,1])$ cannot be embedded in $\mathbb{R}^n$ for $n > 3$. Likewise, in [18], it is shown that $F_n([0,1]^2)$ and $F_2([0,1]^n)$ cannot be embedded in $\mathbb{R}^{2n}$ for $n \geq 3$. Wu, in [27], proved that, for n odd, $F_n(S^1)$ has the homology of S^n and, for n even, $H^0(F_n(S^1)) = H^{n-1}(F_n(S^1)) = \mathbb{Z}$, and $H^i(F_n(S^1)) = 0$ if $i \neq 0, n-1$. Schori, in [26], showed that for a 2-manifold M, $F_2(M)$ is a 4-manifold. In [12], Illanes saw that if X is a locally connected normal space then $F_n(X)$ is unicoherent for $n \geq 3$. Macías proved in [15] that if X is a continuum then $F_n(X)$ is unicoherent for $n \geq 3$. He proves that $\check{H}^1(F_n(X), \mathbb{Z}) = 0$.

In this paper we provide techniques which allow us to compute the Euler characteristics of the n-symmetric products of finite dimensional manifolds. Specifically, this is the aim of Section 3, where we make the explicit computation for X an orientable compact surface without boundary.

In a final remark we suggest the possibility of using our techniques to study the dynamics of certain hyperbolic dynamical systems, such as the G-horseshoe.

2. Definitions and preliminary results

From now on, X will denote a locally compact, metric ANR. Let $U \subset X$ be an open set. By a semidynamical system we mean a locally defined continuous map $f : U \to X$.

We say that a function $\sigma : \mathbb{Z} \to X$ is a *solution to f through* x in $N \subset U$ if $f(\sigma(i)) = \sigma(i+1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The *invariant part* of N, $Inv(N, f)$, is defined as the set of all $x \in N$ that admit a solution to f through x in N.

A compact set $K \subset U$ is *invariant* if $f(K) = K$. An invariant compact set K is *isolated with respect to* f if there exists a compact neighborhood N of K such that $Inv(N, f) = K$. The neighborhood N is called an *isolating neighborhood* of K.

The *n-symmetric product* of X, $F_n(X)$, is the closed subspace of 2^X, endowed with the Hausdorff metric, consisting of all non-empty subsets of X with at most n points.

A semidynamical system $f : U \to X$ induces in a natural way another one, $F_n(f) : F_n(U) \to F_n(X)$.

Let $K \subset U$ be a compact isolated invariant set and let N be any isolating neighborhood of K. Consider an open set W such that $K \subset W \subset N$. Take $F_n(f)|_{F_n(W)} : F_n(W) \to F_n(X)$. It is clear that $Fix(F_n(f)|_{F_n(W)}) \subset F_n(K)$; then $Fix(F_n(f)|_{F_n(W)})$ is a compact subset of $F_n(W)$. On the other hand, $F_n(f)|_{F_n(W)}$ is a compact map because it admits an obvious extension to $F_n(N)$.

The set $F_n(W)$ is an open subset of $F_n(X)$ and, since X is an ANR, $F_n(X)$ is an ANR for all $n \in \mathbb{N}$ ([19]).

Then, $i_{F_n(X)}(F_n(f)|_{F_n(W)}, F_n(W))$, the fixed point index of $F_n(f)|_{F_n(W)}$ in $F_n(W)$, is well defined. For information about the fixed point index theory, the reader is referred to [9], [20], [21], and [11].

It would be interesting to study the fixed point index in the so-called n-symmetric products, $SP_n(X)$, constructed as the quotient of X^n by the action of the group of permutations of n elements. Let us observe that $F_n(X) = SP_n(X)$ if $n \leq 2$. One can expect a better additive behavior of this fixed point index than in the case of $F_n(X)$. In this sense Masih and Rallis, in [16], [17] and [22], constructed certain indices for maps $X \to SP_n(X)$. For more information about these spaces and their relation with algebraic topology, see [1].

Definition 1. We define the *fixed n-finite set index of the pair* (K, f) as

$$I_X^{F_n}(K, f) = i_{F_n(X)}(F_n(f)|_{F_n(W)}, F_n(W)).$$

The condition that K be isolated is sufficient, but not necessary, to guarantee the consistency of this fixed point index.

Remark 1. From the excision property of the fixed point index we have that $I_X^{F_n}(K, f)$ does not depend on the choice of the isolating neighborhood N of K and the open set W.

Remark 2. The spaces $F_n(X)$ are not *growth hyperspaces* of X (see [6]). A compactum B can be locally connected and $F_n(B) \notin$ ANR. So the techniques of [23] for computing the fixed point index in hyperspaces will not be useful in the case of $F_n(X)$.

The main properties of our index follow immediately from the corresponding properties of the fixed point index. They are stated in the following propositions.

Proposition 1 (Ważewski property). $I_X^{F_n}(K, f) \neq 0$ *implies that*

$$K \supset Fix(F_n(f)|_{F_n(W)}) \neq \emptyset.$$

So there exists a periodic orbit of f in K of period $\leq n$.

Proposition 2 (Particular cases of the additivity property). *Let K be a compact isolated invariant set. If K is the disjoint union of two compact isolated invariant sets K_1 and K_2, then*

$$I_X^{F_1}(K, f) = I_X^{F_1}(K_1, f) + I_X^{F_1}(K_2, f)$$

and

$$I_X^{F_2}(K,f) = I_X^{F_2}(K_1,f) + I_X^{F_2}(K_2,f) + I_X^{F_1}(K_1,f)I_X^{F_1}(K_2,f).$$

The proof of the second equality follows from the fact that $F_2(U_1 \cup U_2)$ is homeomorphic to the disjoint union $F_2(U_1) \vee F_2(U_2) \vee (F_1(U_1) \times F_1(U_2))$ for U_1, U_2 disjoint open neighborhoods of K_1 and K_2 respectively.

Proposition 3 (Commutativity property)**.** *Let X, Y be locally compact metric ANRs with U, V open subsets of X and Y respectively. Let*

$$\varphi : U \to Y,$$
$$\psi : V \to X$$

be locally defined maps. Consider $f = \psi \circ \varphi$ and $g = \varphi \circ \psi$. If $K \subset X$ is a compact isolated invariant set with respect to f, then $\varphi(K)$ is a compact isolated invariant set with respect to g and $I_X^{F_n}(K,f) = I_Y^{F_n}(\varphi(K),g)$.

Proposition 4 (Homotopy invariance property)**.** *Let $f : U \times \Lambda \to X$ be a map such that U is an open subset of X and $\Lambda \subset \mathbb{R}$ is a compact interval. Assume that N is an isolating neighborhood for each map $f_\lambda : U \to X$. Then $I_X^{F_n}(Inv(N, f_\lambda), f_\lambda)$ does not depend on $\lambda \in \Lambda$.*

Let us consider a local homeomorphism of the plane, f, with $K = \{p\}$ a non-attracting and non-repelling fixed point. The next results allow us to relate the indices of the iterations of f and the corresponding indices in the symmetric product.

Theorem 1 ([24])**.** *Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism with $p \in U$ a non-attracting and non-repelling fixed point of f such that $\{p\}$ is an isolated invariant set. Then there are a disc D, containing a neighborhood V of p, a finite subset $\{q_1, \ldots, q_m\} \subset D$ and a map $\overline{f} : D \to D$ such that $\overline{f}|_V = f|_V$, $\overline{f}(\{q_1, \ldots, q_m\}) \subset \{q_1, \ldots, q_m\}$, and for every $k \in \mathbb{N}$, $Fix((\overline{f})^k) \subset \{p, q_1, \ldots, q_m\}$.*

Moreover,

a) *(Le Calvez-Yoccoz,* [14]*). If f is orientation-preserving, then*

$$i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N}, \\ 1 & \text{if } k \notin r\mathbb{N}, \end{cases}$$

where $k \in \mathbb{N}$, q is the number of periodic orbits of $\overline{f}$ (excluding p) and r is their period.

b) *If f is orientation-reversing, then there are integers $\delta \in \{0,1,2\}$ and q such that*

$$i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - \delta & \text{if } k \text{ is odd}, \\ 1 - \delta - 2q & \text{if } k \text{ is even}, \end{cases}$$

where q is the number of orbits of period 2 and δ is the number of fixed points of $\overline{f}$ in $\{q_1, \ldots, q_m\}$, and there is no other orbit of $\overline{f}$ in $\{q_1, \ldots, q_m\}$.

If R is a finite set of r elements, let

$$C_s^r = Card(\{S \subset R : Card(S) = s\}).$$

A consequence of the above theorem is the following proposition. The reader can find its proof in Section 4.

Proposition 5. *Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, with $p \in U$ a non-attracting and non-repelling fixed point of f such that $\{p\}$ is an isolated invariant set.*

a) *If f is orientation-preserving, q is the number of periodic orbits of $\overline{f}$ in $\{q_1, \dots, q_m\}$ and r is their period, then for every $n \in \mathbb{N}$*

$$1 = \sum_{1 \le jr \le n} C_j^q + \sum_{0 \le jr < n} C_j^q I_D^{F_{n-jr}}(\{p\}, \overline{f}).$$

b) *If f is orientation-reversing, q the number of period-two orbits of $\overline{f}$ in $\{q_1, \dots, q_m\}$ and $q' \le 2$ the number of fixed points of $\overline{f}$ in $\{q_1, \dots, q_m\}$, then for every $n \in \mathbb{N}$*

$$1 = \sum_{\substack{1 \le 2j+j' \le n \\ j, j' \ge 0}} C_j^q C_{j'}^{q'} + \sum_{\substack{0 \le 2j+j' < n \\ j, j' \ge 0}} C_j^q C_{j'}^{q'} I_D^{F_{n-(2j+j')}}(\{p\}, \overline{f}).$$

Remark 3. In case a) of the above proposition, since $\overline{f}$ is locally constant in $\{q_1, \dots, q_m\}$ (see [24]), we have

$$I_D^{F_n}(\{p\}, \overline{f}) = \frac{\sum_{j=1}^n i_{\mathbb{R}^2}(f^j, p)}{n} = \begin{cases} 1 & \text{if } n < r, \\ 1 - q & \text{if } n = r. \end{cases}$$

Moreover, $I_D^{F_n}(\{p\}, \overline{f}) = I_D^{F_{kr+1}}(\{p\}, \overline{f})$ for every $n \in (kr, (k+1)r)$.

3. The Euler characteristic of the n-symmetric product of a manifold

The aim of this section is to develop techniques which allow us to compute the Euler characteristic of the n-symmetric product of a finite dimensional manifold X. We will restrict ourselves to the case when X is an orientable, compact surface without boundary. This setting will provide us with techniques to study the general case.

If we choose an adequate dynamical system (homeomorphism) $F : X \to X$ ($F \simeq id$), the Euler characteristic of $F_n(X)$ is

$$\chi(F_n(X)) = \Lambda(F_n(id)) = \Lambda(F_n(F)) = i_{F_n(X)}(F_n(F), F_n(X)),$$

and, if F is such that the number of its periodic orbits of period $\le n$ is finite, by the additivity property, we only have to compute a finite number of indices $i_{F_n(X)}(F_n(F), \bigcup_{j=1}^r \overline{\alpha}_{p_j}^j)$ for $\overline{\alpha}_{p_j}^j$ periodic orbits of F of period p_j with $\sum_{j=1}^r p_j \le n$. The above fixed point indices, denoted by $i_n(F, \bigcup_{j=1}^r \overline{\alpha}_{p_j}^j)$, are defined in small enough neighborhoods, in $F_n(X)$, of the isolated fixed points $\bigcup_{j=1}^r \overline{\alpha}_{p_j}^j$.

Note that if $f : X \to X$ is a diffeomorphism of a manifold X of dimension m with p a hyperbolic fixed point for f, then by the Grobman-Hartman theorem (see [10]) we can reduce the study of $I_X^{F_n}(\{p\}, f)$ to the linear case $I_{\mathbb{R}^m}^{F_n}(\{0\}, Df(p))$.

Let U be an open neighborhood of $\{0\}$ in $\mathbb{R}^m$ and let $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ be a linear map. Assume that $K = \{0\}$ is a compact isolated invariant set. The study of the index $I_{\mathbb{R}^m}^{F_n}(\{0\}, f)$ gives information which allows us to calculate $\chi(F_n(X))$ for a compact manifold X.

Let us denote by $D(\lambda_1, \dots, \lambda_m)$ the diagonal $m \times m$ matrix with $\lambda_1, \dots, \lambda_m$ on the diagonal.

The only linear cases which we will need here are given in the next proposition:

Proposition 6.

$$I^{F_n}_{\mathbb{R}^m}(\{0\}, D(0,\dots,0)) = 1$$

and

$$I^{F_n}_{\mathbb{R}^m}(\{0\}, D(2,\dots,2)) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ -1 & \text{if } m \text{ and } n \text{ are odd,} \\ 0 & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

The first equality is trivial, and the second one is proved in the Appendix.

The next theorem provides a complete study of $I^{F_n}_{\mathbb{R}^m}(\{0\}, f)$ for f a linear map and 0 a hyperbolic fixed point. We give an outline of the proof in the Appendix (see [25] for a complete proof).

This result is useful if one wants to study the Euler characteristic of the symmetric product of a manifold of dimension $n > 2$.

Theorem 2. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ be a linear map with $K = \{0\}$ a compact isolated invariant set. Consider the set of the real eigenvalues (repeated) which have modulus greater than 1, $\{\lambda_1, \dots, \lambda_r\}$.*

Let r_2 be the number of eigenvalues greater than 1, and r_{-2} the number of eigenvalues smaller than -1. Of course $r = r_2 + r_{-2}$. Then,

$$I^{F_n}_{\mathbb{R}^m}(\{0\}, f) = \begin{cases} \text{if } r_2 \text{ is odd and } r_{-2} \text{ is even,} \\ I^{F_n}_{\mathbb{R}}(\{0\}, D(2)) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd;} \end{cases} \\ \text{if } r_2 \text{ is even and } r_{-2} \text{ is odd,} \\ I^{F_n}_{\mathbb{R}}(\{0\}, D(-2)) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^k & \text{if } n = 2k+1; \end{cases} \\ \text{if } r_2 \text{ is odd and } r_{-2} \text{ is odd,} \\ I^{F_n}_{\mathbb{R}^2}(\{0\}, D(2,-2)) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd;} \end{cases} \\ \text{if } r_2 \text{ is even and } r_{-2} \text{ is even,} \\ I^{F_n}_{\mathbb{R}^r}(\{0\}, D(0,\dots,0)) = 1. \end{cases}$$

From now on, we study $\chi(F_n(X))$ for X an orientable, compact surface without boundary.

In the next proposition we compute $\chi(F_n(S^k))$.

Proposition 7. *The Euler characteristic $\chi(F_n(S^k))$ of the n-symmetric products of S^k is*

$$\chi(F_n(S^{2k+1})) = 0$$

for all $n \in \mathbb{N}$, and

$$\chi(F_n(S^{2k})) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \geq 2. \end{cases}$$

Proof. Consider the dynamical system $J : S^k \to S^k$, shown in Figure 1.

We have $J \simeq id$, and there are two hyperbolic fixed points, a repeller p and an attractor q.

We have

$$\chi(F_n(S^k)) = \Lambda(F_n(J)) = I^{F_n}_{S^k}(S^k, J) = i_n(J, p) + i_n(J, q) + i_n(J, \{p, q\}).$$

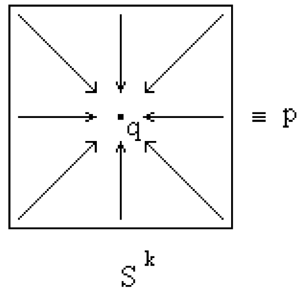

FIGURE 1.

Let us consider a small enough open neighborhood of q, $U_n(q)$, in $F_n(S^k)$. Since q is an attractor, we can construct a homotopy $H : cl(U_n(q)) \times I \to F_n(S^k)$ such that $H_0 = F_n(J)$ and $H_1 \equiv q$ and $H(\bar{x}, t) \neq \bar{x} \quad \forall (\bar{x}, t) \in \partial(U_n(q)) \times I$. Then, by the homotopy property of the fixed point index, it is obvious that

$$i_n(J, q) = i_{F_n(S^k)}(H_0, U_n(q)) = i_{F_n(S^k)}(H_1, U_n(q)) = 1.$$

The next step is to prove that $i_n(J, \{p, q\}) = i_{n-1}(J, p)$. Given the open balls $V_1 = B(p, \epsilon)$ and $V_2 = B(q, \epsilon)$, we define the open neighborhood of the point $\{p, q\} \in F_n(S^k)$,

$$U_n(\{p, q\}) = \{\bar{x} \in F_n(S^k) : \bar{x} \subset \bigcup V_i \text{ and } \bar{x} \cap V_i \neq \emptyset \text{ for all } i = 1, 2\}.$$

Given $\bar{x} = \{x_1, \ldots, x_s, x_{s+1}, \ldots, x_t\} \in U_n(\{p, q\})$, with $\{x_1, \ldots, x_s\} \subset V_1$ and $\{x_{s+1}, \ldots, x_t\} \subset V_2$, we define the continuous map

$$F : U_n(\{p, q\}) \to F_{n-1}(S^k)$$

as $F(\{x_1, \ldots, x_t\}) = \{J(x_1), \ldots, J(x_s)\}$.

In the same way we consider the continuous map

$$G : U_{n-1}(p) \to F_n(S^k)$$

defined as $G(\{x_1, \ldots, x_t\}) = \{x_1, \ldots, x_t, q\}$.

Now, we take the compositions $F \circ G : U_{n-1}(p) \to F_{n-1}(S^k)$ and $G \circ F : F^{-1}(U_{n-1}(p)) \to F_n(S^k)$. It is obvious that $F \circ G = F_{n-1}(J)$. On the other hand, $(G \circ F)(\{x_1, \ldots, x_t\}) = \{J(x_1), \ldots, J(x_s), q\}$, and it is not difficult to construct a homotopy

$$H : cl(F^{-1}(U_{n-1}(p))) \times I \to F_n(S^k)$$

such that $H_0 = F_n(J)$ and $H_1 = G \circ F$, with

$$H(\bar{x}, t) \neq \bar{x} \quad \text{for all } (\bar{x}, t) \in \partial(F^{-1}(U_{n-1}(p))) \times I.$$

Then, using the commutativity and the homotopy properties of the fixed point index, we have that

$$\begin{aligned} i_n(J, \{p, q\}) &= i_{F_n(S^k)}(F_n(J), U_n(\{p, q\})) = i_{F_n(S^k)}(G \circ F, F^{-1}(U_{n-1}(p))) \\ &= i_{F_{n-1}(S^k)}(F \circ G, U_{n-1}(p))) = i_{F_{n-1}(S^k)}(F_{n-1}(J), U_{n-1}(p))) = i_{n-1}(J, p). \end{aligned}$$

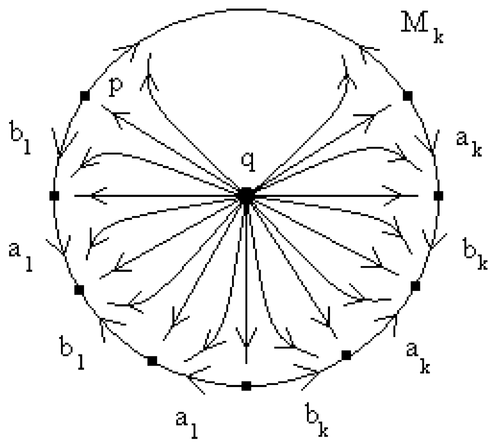

FIGURE 2.

Then $\chi(F_n(S^k)) = i_n(J,p) + 1 + i_{n-1}(J,p)$, and from Proposition 6 and the Grobman-Hartman theorem, we have

$$i_n(J,p) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ -1 & \text{if } k \text{ and } n \text{ are odd,} \\ 0 & \text{if } k \text{ is odd and } n \text{ is even.} \end{cases}$$

Now the result follows automatically. □

Remark 4. Let us notice that we can construct a map $F_k : S^{2k+1} \to S^{2k+1}$ homotopic to the identity without periodic points. The equality $\chi(F_n(S^{2k+1})) = 0$ follows from this fact. We define F_k as the restriction to $S^{2k+1} \subset C^{k+1}$ of the map $(z_1, \ldots, z_{k+1}) \mapsto (e^{2i\pi\alpha} z_1, \ldots, e^{2i\pi\alpha} z_{k+1})$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

We can also use this map to compute $\chi(F_n(S^{2k+2}))$. In fact, let us consider $g : [-1,1] \to [-1,1]$ with $g(x) = 2x$ if $|x| \le 1/2$, and $g(x) = \frac{x}{|x|}$ if $1/2 \le |x| \le 1$. The map

$$F_k \times g : S^{2k+1} \times [-1,1] \to S^{2k+1} \times [-1,1]$$

defines a continuous map on the sphere S^{2k+2} obtained by identifying each sphere $S^{2k+1} \times \{\epsilon\}$, $\epsilon \in \{-1,1\}$, to a point. The only periodic orbits are the two fixed points, where the map is locally constant.

The same ideas can be applied to the torus T, to prove that $\chi(F_n(T)) = 0$.

Let us compute the Euler characteristic of the n-symmetric product of the compact oriented surfaces of genus k, $\chi(F_n(M_k))$.

Proposition 8. *The Euler characteristic of $F_n(M_k)$, with $k \ge 2$, is*

$$\chi(F_n(M_k)) = \sum_{j=1}^{n} (-1)^j C_j^{2k-3+j}.$$

If $k = 1$, then $\chi(F_n(T)) = 0$.

Proof. Let us consider the dynamical system $J : M_k \to M_k$ shown in Figure 2.

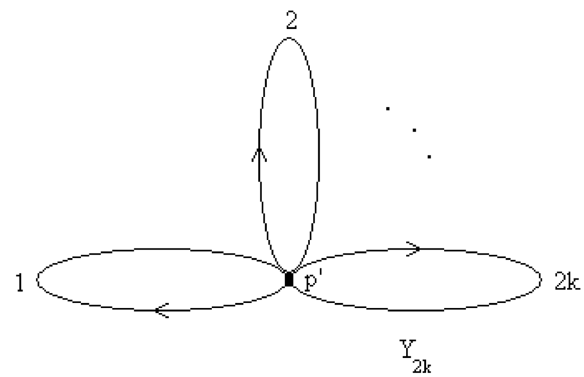

FIGURE 3.

We have that $J \simeq id$, with two fixed points p and q. The point q is a source and the map $J|_{M_k \setminus \{q\}}$ is conjugated to the product

$$L_{2k} \times f : Y_{2k} \times [0,1) \to Y_{2k} \times [0,1),$$

where $f(x) = x^2$ and $L_{2k} : Y_{2k} \to Y_{2k}$ is the dynamical system defined on the pointed union of $2k$ loops Y_{2k} shown in Figure 3.

The Euler characteristic of $F_n(M_k)$ is

$$\chi(F_n(M_k)) = \Lambda(F_n(J)) = I^{F_n}_{M_k}(M_k, J) = i_n(J,p) + i_n(J,q) + i_n(J,\{p,q\}).$$

By Proposition 6 and the Grobman-Hartman theorem, we have $i_n(J,q) = 1$.

On the other hand, let us see that $i_n(J,\{p,q\}) = i_{n-1}(J,p)$. We consider the continuous maps

$$F : U_n(\{p,q\}) \to F_{n-1}(M_k) \text{ and } G : U_{n-1}(p) \to F_n(M_k)$$

defined as in the proof of Proposition 7, and a homotopy

$$H : cl(F^{-1}(U_{n-1}(p))) \times I \to F_n(M_k)$$

such that $H_0 = F_n(J)$ and $H_1 = G \circ F$, with

$$H(\bar{x},t) \neq \bar{x} \quad \text{for all } (\bar{x},t) \in \partial(F^{-1}(U_{n-1}(p))) \times I$$

(for a construction of the homotopy H, see the proof of Proposition 6 in the Appendix).

From the commutativity and the homotopy invariance properties of the fixed point index, we have $i_n(J,\{p,q\}) = i_{n-1}(J,p)$, and therefore

$$\chi(F_n(M_k)) = i_n(J,p) + 1 + i_{n-1}(J,p).$$

It only remains to compute $i_n(J,p)$. Since $J|_{M_k \setminus \{q\}}$ is conjugated to $L_{2k} \times f$, then $i_n(J,p) = i_n(L_{2k} \times f, (p',0)) = i_n(L_{2k}, p')$. The last equality follows from the homotopy and commutativity properties of the fixed point index.

Let us define the dynamical systems $H_k, H'_k : Z_k \to Z_k$ with Z_k the union of k arcs connected by the endpoints (see Figure 4).

Given a fixed point $\bar{\alpha}$ of $F_n(H_k)$, we denote $i_{F_n(Z_k)}(F_n(H_k), \bar{\alpha}) = i_n(H_k, \bar{\alpha})$.

Let us prove that $i_n(L_{2k}, p') = i_n(H_{2k}, p)$. Given the map $g : [0,1] \to [0,1]$ with $g(x) = 2x$ if $|x| \leq 1/2$, and $g(x) = \frac{x}{|x|}$ if $1/2 \leq |x| \leq 1$, the restriction of L_{2k} to

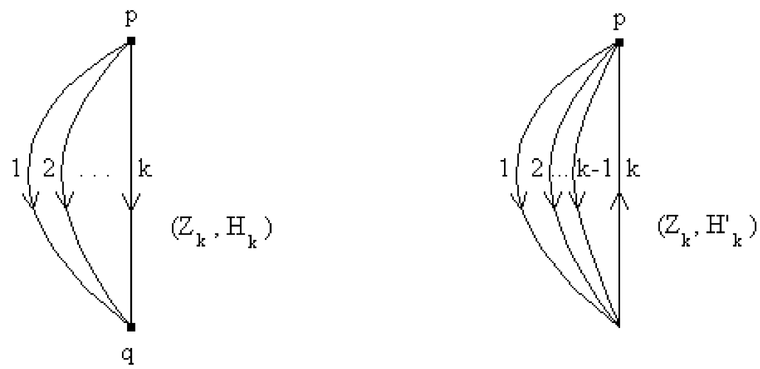

FIGURE 4.

each loop can be considered as a map of the type

$$g : [0,1]/(0 \equiv 1) \to [0,1]/(0 \equiv 1).$$

We can consider the dynamical system $L_{2k} : Y_{2k} \to Y_{2k}$ as a identification in $H_{2k} : Z_{2k} \to Z_{2k}$ of the points p and q to a point p'. If $x \in Z_{2k}$, we call $[x] \in Y_{2k}$ the corresponding point obtained by the identification.

Given a small enough neighborhood $U_n(p')$ of p' in $F_n(Y_{2k})$, let

$$\bar{x} = \{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\} \in U_n(p')$$

with $\{[x_1], \ldots, [x_r]\}$ the points of $\bar{x}$ contained in the local repelling part of p' in Y_{2k}.

Then let us consider the map $F : U_n(p') \subset F_n(Y_{2k}) \to Z_{2k}$ defined as

$$F(\{[x_1], \ldots, [x_r], [x_{r+1}], \ldots, [x_s]\}) = \{H_{2k}(x_1), \ldots, H_{2k}(x_r), p\}$$

If $r = s$, the point p does not appear in the image of F.

Let $G : U_n(p) \subset F_n(Z_{2k}) \to F_n(Y_{2k})$ be the map defined as

$$G(\{x_1, \ldots, x_r\}) = \{[x_1], \ldots, [x_r]\}$$

By the commutativity property of the fixed point index applied to F and G we obtain that $i_n(L_{2k}, p') = i_n(H_{2k}, p)$. Therefore

$$i_n(J, p) = i_n(H_{2k}, p),$$

and we only have to compute $i_n(H_{2k}, p)$.

If $n \geq 2$, we have

$$\begin{aligned} I_{Z_k}^{F_n}(Z_k, H_k) &= i_n(H_k, p) + i_n(H_k, q) + i_n(H_k, \{p, q\}) \\ &= i_n(H_k, p) + 1 + i_{n-1}(H_k, p). \end{aligned}$$

The equality $i_n(H_k, q) = 1$ is a consequence of the fact that q is an attractor, and $i_n(H_k, \{p, q\}) = i_{n-1}(H_k, p)$ follows again from the homotopy invariance and the commutativity properties of the fixed point index.

Using similar arguments it is easy to see that

$$I_{Z_k}^{F_n}(Z_k, H'_k) = i_n(H_{k-1}, p).$$

Since $H_k \simeq H'_k$, then $I^{F_n}_{Z_k}(Z_k, H_k) = I^{F_n}_{Z_k}(Z_k, H'_k)$ and

$$i_n(H_{k-1}, p) = i_n(H_k, p) + 1 + i_{n-1}(H_k, p). \tag{1}$$

This formula allows us to compute $i_n(H_k, p)$ in a recurrent way (it is easy to see that $i_n(H_1, p) = 0$ for all n). Our aim is to obtain $i_n(H_k, p)$ in an explicit expression by an induction argument.

Let us prove that $i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^n C_n^{k+n-2}$.

Let $n = 2$ and $k = 1$. Then, since $i_n(H_1, p) = 0$, we have $i_2(H_1, p) - i_1(H_1, p) = (-1)^2 C_2^1 = 0$. Let us suppose that

$$i_n(H_k, p) - i_{n-1}(H_k, p) = (-1)^n C_n^{k+n-2}$$

for all $n \geq 2, k \geq 1$ with $n + k \leq m_0$, and consider n, k with $n + k = m_0 + 1$. Then, using (1), we have

$$\begin{aligned} i_n(H_{k-1}, p) &= i_n(H_k, p) + i_{n-1}(H_k, p) + 1, \\ -i_{n-1}(H_{k-1}, p) &= -i_{n-1}(H_k, p) - i_{n-2}(H_k, p) - 1. \end{aligned}$$

It follows that

$$\begin{aligned} & i_n(H_k, p) - i_{n-1}(H_k, p) \\ &\quad = i_n(H_{k-1}, p) - i_{n-1}(H_{k-1}, p) + i_{n-2}(H_k, p) - i_{n-1}(H_k, p) \\ &\quad = (-1)^n C_n^{k+n-3} + (-1)^n C_{n-1}^{k+n-3} = (-1)^n C_n^{k+n-2}, \end{aligned}$$

and the result is proved.

In the same way, it follows that $i_1(H_k, p) = -(k-1)$, and then

$$i_n(H_k, p) = \sum_{j=1}^{n} (-1)^j C_j^{k-2+j}$$

and

$$\begin{aligned} \chi(F_n(M_k)) &= i_n(H_{2k}, p) + i_{n-1}(H_{2k}, p) + 1 \\ &= i_n(H_{2k-1}, p) = \sum_{j=1}^{n} (-1)^j C_j^{2k-3+j}. \end{aligned}$$

□

Remark 5. Given a manifold X and a continuous map $F : X \to X$, we can obtain, under certain conditions of hyperbolicity, information about the dynamics of F by studying the fixed point indices $I^{F_n}_X(Inv(X, F), F)$. Certainly, there are other techniques which allow us to study this, but it seemed interesting for us to present this alternative method.

Example. Dynamics of the G-horseshoe. If we want to study the periodic orbits of the G-horseshoe with our techniques, let us consider the dynamical system $F : C \to C$ given by the extended G-horseshoe of Figure 5. We are interested in detecting the periodic orbits of F on I^2 (the unique periodic orbit out of I^2 is a fixed point). Let us consider the continuous map $g : \Pi \circ F|_{S^1} : S^1 \to S^1$ defined as the composition of $F|_{S^1}$ with the projection $\Pi : C \to S^1$, where S^1 is the interior circle of C. It is not difficult to see, by the homotopy invariance and the commutativity properties of the fixed point index, that

$$I^{F_n}_C(Inv(I^2, F), F) = I^{F_n}_{S^1}(Inv(I, g), g). \tag{2}$$

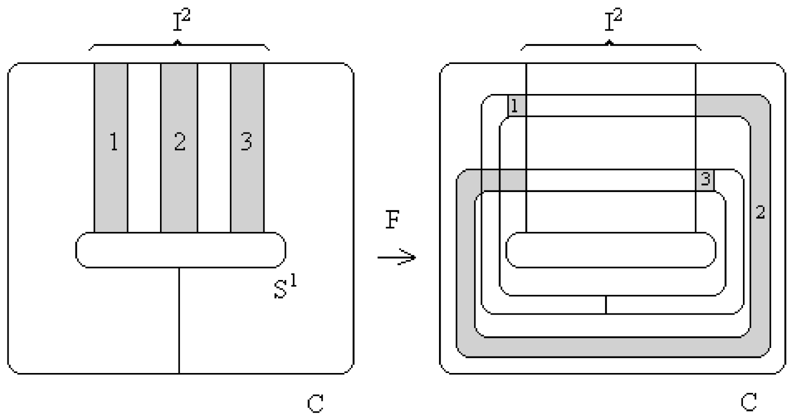

FIGURE 5.

Let us observe that $g|_I$ is an expansion, and $F|_{I^2}$ is a contraction in the vertical direction and an expansion in the horizontal one. Then, given n fixed, the number of periodic orbits of period $\leq n$ for g and F is finite, and the fixed points of $F_n(g)$ and $F_n(F)$ are isolated.

It is not hard to prove that if $\bar{\alpha}$ is a periodic orbit of period n of $F|_{I^2}$, then $i_{F_n(C)}(F_n(F), \bar{\alpha}) = -1$ (the same fact occurs with $g|_I$). Then we can see, using (2) and an induction argument, that the number of periodic orbits of all periods of $F|_{I^2}$ is the same as in the case of $g|_I$.

By the commutativity and the homotopy properties of the fixed point index,

$$I_C^{F_n}(Inv(C,F),F) = I_{S^1}^{F_n}(Inv(S^1,g),g) = I_{S^1}^{F_n}(Inv(S^1,f),f),$$

where $f : S^1 \to S^1$ is the doubling angle map. A careful observation of f and g allows us to see that, although f has one fixed point and $g|_I$ has two (repelling) fixed points, the remaining periodic orbits are the same in both dynamical systems.

Since the set $\{x \in S^1 : f^n(x) = x\}$ has $2^n - 1$ points, then the set $\{x \in I^2 : F^n(x) = x\}$ has 2^n points. So, we have a characterization of the periodic orbits of the G-horseshoe.

4. Appendix. Proofs

Proof of Proposition 5. Let us see the proof of a) (the proof of b) is analogous). Since D is an AR, $1 = I_D^{F_n}(D, \overline{f})$.

Let us consider the point $\overline{\alpha(l)} = \overline{\alpha}_1 \cup \cdots \cup \overline{\alpha}_l \in F_n(D)$ with $\overline{\alpha}_i = \{q_{i_1}, \ldots, q_{i_r}\}$ a periodic orbit of $\overline{f}$ in $\{q_1, \ldots, q_m\}$ for all $i = 1, \ldots, l$.

$Per(\overline{f})$ is the set of periodic orbits of $\overline{f}$ in $\{q_1, \ldots, q_m\}$. Let us denote

$$i_{F_n(D)}(F_n(\overline{f}), \overline{\alpha(j)}) = i_n(\overline{f}, \overline{\alpha(j)}).$$

From the additivity property of the fixed point index for ANRs, we have

$$1 = I_D^{F_n}(D, \overline{f}) = \sum_{\substack{\overline{\alpha(j)} \subset Per(\overline{f}) \\ jr \leq n}} i_n(\overline{f}, \overline{\alpha(j)}) + \sum_{\substack{\overline{\alpha(j)} \subset Per(\overline{f}) \\ jr < n}} i_n(\overline{f}, p \cup \overline{\alpha(j)}) + i_n(\overline{f}, p).$$

Since $\overline{f}$ is locally constant in each q_i, see [24], we have

$$i_n(\overline{f}, \overline{\alpha(j)}) = 1$$

for all $\overline{\alpha(j)} \subset Per(\overline{f})$, $jr \leq n$.

Let $\overline{\alpha(j)}$ be fixed with $jr < n$. We prove that

$$i_n(\overline{f}, p \cup \overline{\alpha(j)}) = i_{n-jr}(\overline{f}, p).$$

Let $U_n(p \cup \overline{\alpha(j)})$ be a small enough neighborhood in $F_n(D)$ of the point $p \cup \overline{\alpha(j)}$, and let $\bar{x} \in U_n(p \cup \overline{\alpha(j)})$ with $\bar{x}_p = \bar{x} \cap B(p, \epsilon)$ for ϵ small enough. The set $\bar{x}_p = \{x_1, \ldots, x_l\}$ is such that $1 \leq l \leq n - jr$.

Let $F : U_n(p \cup \overline{\alpha(j)}) \to F_{n-jr}(D)$ and $G : U_{n-jr}(p) \to F_n(D)$ be the continuous maps

$$F(\bar{x}) = \{\overline{f}(x_1), \ldots, \overline{f}(x_l)\}, \quad G(\bar{x}) = \overline{\alpha(j)} \cup \bar{x}.$$

The map $F \circ G : U_{n-jr}(p) \to F_{n-jr}(D)$ is such that

$$(F \circ G)(\bar{x}) = F_{n-jr}(\overline{f})(\bar{x}).$$

On the other hand, since $\overline{f}$ is locally constant in each $q_i \in \{q_1, \ldots, q_m\}$, the map $G \circ F : F^{-1}(U_{n-jr}(p)) \to F_n(D)$ is such that

$$(G \circ F)(\bar{x}) = F_n(\overline{f})(\bar{x}).$$

From the commutativity property of the fixed point index for ANRs we have the equality

$$i_n(\overline{f}, p \cup \overline{\alpha(j)}) = i_{n-jr}(\overline{f}, p).$$

The proof of case a) is finished. □

Proof of Proposition 6. Let us see that $I_{\mathbb{R}^m}^{F_n}(\{0\}, 2Id) = 1$ for $m = 2$ (the case of m even will be analogous).

Let $U_0 = B(0, 1)$ be an open neighborhood of $\{0\}$ and let $H : F_n(cl(U_0)) \times I \to F_n(\mathbb{R}^2)$ be the homotopy

$$H(\{x_1, \ldots, x_r\}, t) = \begin{cases} \{A(t)(2x_1), \ldots, A(t)(2x_r)\} & \text{if } t \in [0, 1/2], \\ \{2(1-t)A(1/2)(2x_1), \ldots, 2(1-t)A(1/2)(2x_r)\} & \text{if } t \in [1/2, 1], \end{cases}$$

with

$$A(t) = \begin{pmatrix} \cos(\frac{2\pi}{n+1} 2t) & \sin(\frac{2\pi}{n+1} 2t) \\ -\sin(\frac{2\pi}{n+1} 2t) & \cos(\frac{2\pi}{n+1} 2t) \end{pmatrix}.$$

We consider $x_i \neq x_j$ if $i \neq j$. It is obvious that $r \leq n$.

The continuity of H is clear, and it is not hard to see that $H(\bar{x}, t) \neq \bar{x}$ for all $(\bar{x}, t) \in \partial(F_n(U_0)) \times I$. Since $H_0 = F_n(2Id)$ and $H_1 = F_n(D(0,0))$, we have proved the result for $m = 2$.

Let us see that

$$I_{\mathbb{R}^m}^{F_n}(\{0\}, 2Id) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd,} \end{cases}$$

for m odd. We will prove the result for $m = 1$ (the general case is easy to obtain by combining the cases $m = 1$ and m even).

Let us consider the map $g : J \to J$ with $g(x) = x^{1/3}$ and $J = [-1, 1]$. The only periodic orbits are the fixed points $\{-1, 0, 1\}$.

Since $F_n(J)$ is an absolute retract, we have

$$I_J^{F_n}(J, g) = \Lambda(F_n(g)) = \Lambda(F_n(id)) = 1.$$

Let us denote $i_{F_n(J)}(F_n(g), \bar{\alpha}) = i_n(g, \bar{\alpha})$ for $\bar{\alpha} \in Fix(F_n(g))$. Then

$$1 = I_J^{F_n}(J, g) = \sum_{\bar{\alpha} \subset \{-1,0,1\}} i_n(g, \bar{\alpha}).$$

Using the commutativity and the homotopy invariance properties of the fixed point index as in the proof of Proposition 7, it is not difficult to see that

$$i_n(g, 1) = i_n(g, -1) = i_n(g, \{-1, 1\}) = 1,$$
$$i_n(g, \{-1, 0\}) = i_n(g, \{0, 1\}) = i_{n-1}(g, 0),$$

and

$$i_n(g, \{-1, 0, 1\}) = i_{n-2}(g, 0).$$

Then, for $n > 2$,

$$1 = I_J^{F_n}(J, g) = i_n(g, 0) + 2i_{n-1}(g, 0) + i_{n-2}(g, 0) + 3.$$

Since $I_{\mathbb{R}}^{F_n}(\{0\}, 2Id) = i_n(g, 0)$, by an induction argument on the last formula we finish the proof. □

Proof of Theorem 2. Since $\{0\}$ is an isolated invariant set, the eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ of f have modulus different from 1. The first equality of the theorem, which reduces the study of the fixed point index in $\mathbb{R}^m$ to the cases of $\mathbb{R}$ and $\mathbb{R}^2$, it is easy to prove by using the techniques employed in the proof of Proposition 6.

On the other hand, the computation of $I_{\mathbb{R}}^{F_n}(\{0\}, D(2))$ follows from studying the dynamical system $g_1 : J \to J$ defined as $g_1(x) = x^{1/3}$ with $J = [-1, 1]$. In fact,

$$1 = \chi(F_n(J)) = I_J^{F_n}(J, g_1) = \sum_{\bar{\alpha} \subset \{-1,0,1\}} i_n(g_1, \bar{\alpha}),$$

and we compute $I_{\mathbb{R}}^{F_n}(\{0\}, D(2)) = i_n(g_1, 0)$ from the above equality. In an analogous way we have $I_{\mathbb{R}}^{F_n}(\{0\}, D(-2))$ with $g_2(x) = -x^{1/3}$.

It only remains to compute $I_{\mathbb{R}^2}^{F_n}(\{0\}, D(2, -2))$. Let us consider the dynamical systems $F = s \circ f : S^2 \to S^2$ and $G = s \circ g : S^2 \to S^2$, where $s : S^2 \to S^2$ is a symmetry with respect to the plane $\{z = 0\}$ and $f, g : S^2 \to S^2$ are the dynamical systems shown in Figure 6.

For the dynamical system given by F, the fixed point p is of type $D(2, -2)$ and q is an attractor. The fixed points a and b of G are of type $D(2, -1/2)$ and $D(-2, 1/2)$ respectively. The pairs $\{c_1, c_2\}$ and $\{d_1, d_2\}$ are attracting periodic orbits of period 2.

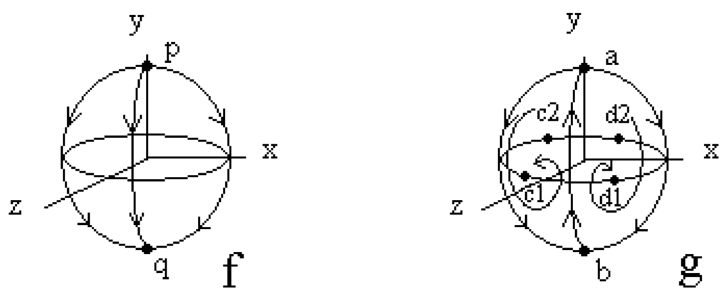

FIGURE 6.

We have $F \simeq G$. Therefore, if $n \geq 2$,

$$\begin{aligned} I_{S^2}^{F_n}(S^2, G) &= I_{S^2}^{F_n}(S^2, F) = i_n(F,p) + i_n(F,q) + i_n(F,\{p,q\}) \\ &= i_n(F,p) + 1 + i_{n-1}(F,p). \end{aligned}$$

Let us prove the equality $I_{S^2}^{F_n}(S^2, G) = 1$. By the additivity property of the fixed point index,

$$I_{S^2}^{F_n}(S^2, G) = \sum_{\bar{\alpha} \subset \{a,b,\{c1,c2\},\{d1,d2\}\}} i_n(G, \bar{\alpha}). \tag{3}$$

We have that $i_n(G,a) = I_{\mathbb{R}}^{F_n}(\{0\}, D(2))$ and $i_n(G,b) = I_{\mathbb{R}}^{F_n}(\{0\}, D(-2))$. The only difficulty is to compute $i_n(G, \{a,b\})$.

Let $J_1 = J_2 = [-1,1]$. We denote by $X = J_1 \vee J_2$ the disjoint union of the intervals. Let us consider the map $h : X \to X$, defined as $h(x) = x^{1/3} \in J_1$ if $x \in J_1$ and $h(x) = -x^{1/3} \in J_2$ if $x \in J_2$.

Since $\chi(F_1(X)) = I_X^{F_1}(X,h) = 2$ and $\chi(F_n(X)) = I_X^{F_n}(X,h) = 3$ if $n > 1$, we can prove that

$$i_n(h, \{0,0\}) = \begin{cases} -1 & \text{if } n = 4k+2, \\ 1 & \text{if } n = 4k+3, \\ 0 & \text{otherwise}, \end{cases}$$

for $k \in \mathbb{N}$.

Since $i_n(h,\{0,0\}) = i_n(G,\{a,b\})$, the equality $I_{S^2}^{F_n}(S^2, G) = 1$ follows from (3).

Then $i_n(F,p) + i_{n-1}(F,p) = 0$. Since $i_1(F,p) = -1$, we obtain the value of $I_{\mathbb{R}^2}^{F_n}(\{0\}, D(2,-2)) = i_n(F,p)$, and the proof is finished. □

ACKNOWLEDGEMENTS

I wish to thank Professor F. R. Ruiz del Portal for many valuable comments, and the referee for his or her careful reading of the manuscript and suggestions.

REFERENCES

[1] M. Aguilar, S. Gitler, C. Prieto, *Topología algebraica: un enfoque homotópico*. McGraw Hill, México 1998; English transl., Springer-Verlag, New York, 2002. MR 2003c:53001
[2] K. Borsuk, *On the third symmetric potency of the circunference*, Fund. Math. 36 (1949) 235-244. MR 12:42a
[3] K. Borsuk, S. Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. 37 (1931) 875-882.
[4] R. Bott, *On the third symmetric potency of* S_1, Fund. Math. 39 (1952) 364-368. MR 14:1003e

[5] R. F. Brown, *The Lefschetz fixed point theorem.* Scott, Foresman and Company (1971) Glenview, Illinois. MR 14:1003e
[6] D. W. Curtis, *Hyperspaces of noncompact metric spaces*. Compositio Math. 40, 2 (1980) 139-152. MR 81c:54009
[7] E. N. Dancer, *Degree theory on convex sets and applications to bifurcation*, in Calculus of variations and partial differential equations. Edited by G. Buttazzo, A. Marino, M. K. V. Murthy. Springer-Verlag, Berlin, Heidelberg 2000 (pages 185-225). MR 2002d:49002
[8] R. L. Devaney, *An introduction to chaotic dynamical systems*, Addison-Wesley, 1987. MR 91a:58114
[9] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology, 4 (1965), 1-8. MR 33:1850
[10] R. W. Easton, *Geometric methods for discrete dynamical systems*, Oxford University Press, 1998. MR 2000e:37002
[11] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Math. France 100 (1972) 209-228. MR 46:8213
[12] A. Illanes, *Multicoherence of symmetric products*, An. Inst. Mat. Univ. Nac. Autónoma México 25 (1985) 11-24. MR 87k:54053a
[13] A. Illanes, S. Macías, S. Nadler, *Symmetric products and Q-manifolds*, Contemporary Math. 246 (1999) 137-141. MR 2001b:54011
[14] P. Le Calvez, J. C. Yoccoz, *Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe.* Annals of Math. 146 (1997) 241-293. MR 99a:58129
[15] S. Macías, *On symmetric products of continua*, Topology and its Applic. 92 (1999) 173-182. MR 2000a:54009
[16] S. Masih, *Fixed points of symmetric product mappings of polyhedra and metric absolute neighborhood retracts*, Fund. Math. 80 (1973) 149-156. MR 51:6783
[17] S. Masih, *On the fixed point index and the Nielsen fixed point theorem of symmetric product mappings*, Fund. Math. 102 (1979)143-158. MR 80f:55002
[18] R. Molski, *On symmetric products*, Fund. Math. 44 (1957) 165-170. MR 19:1186e
[19] N. To Nhu, *Investigating the ANR-property of metric spaces*, Fund. Math. 124 (1984), 243-254. MR 86d:54018
[20] R. D. Nussbaum, *Generalizing the fixed point index*, Math. Ann. 228 (1977), 259-278. MR 55:13461
[21] R. D. Nussbaum, *The fixed point index and some applications*, Séminaire de Mathématiques supérieures, vol. 94, Les presses de L'Université de Montréal, 1985. MR 87a:47085
[22] N. Rallis, *A fixed point index theory for symmetric product mappings*, Manuscripta Math. 44 (1983) 279-308. MR 85g:55005
[23] F. R. Ruiz del Portal, J. M. Salazar, *Fixed point index in hyperspaces: A Conley-type index for discrete semidynamical systems*, J. London Math. Soc. (2) 64 (2001) 191-204. MR 2002e:54023
[24] F. R. Ruiz del Portal, J. M. Salazar, *Fixed point index of iterations of local homeomorphisms of the plane: A Conley index approach*, Topology 41 (2002) 1199-1212. MR 2003f:37022
[25] J. M. Salazar, *Indice de punto fijo en hiperespacios e índice de Conley*, Thesis (2001) Universidad Complutense de Madrid.
[26] R. M. Schori, *Hyperspaces and symmetric products of topological spaces*, Fund. Math. 63 (1968) 77-88. MR 38:661
[27] W. Wu, *Note sur les produits essentiels symétriques des espaces topologiques*, C. R. Acad. Sci. Paris 224 (1947) 1139-1141. MR 8:479g

Departamento de Matemáticas, Universidad de Alcalá, Alcalá de Henares, Madrid 28871, Spain
E-mail address: josem.salazar@uah.es